PERIODIC ONE-PARAMETER GROUPS IN THREE-SPACE*

BY

DEANE MONTGOMERY† AND LEO ZIPPIN

Introduction. We shall consider a one-parameter group of homeomorphisms of euclidean three-space, here designated by E, into itself. We shall postulate for our group a certain weak periodic character with respect to the points of E and shall show that it is closely related to the rotation group of three-space. In particular it will be easy to determine when it is topologically equivalent to that group. In anticipation of the explanatory remarks of the next section, we may formulate our principal theorem as follows:

A continuous one-parameter group $T(x; t)$,

$$x \epsilon E, \qquad -\infty < t < \infty,$$

of pointwise-periodic homeomorphisms of E into itself, whose period function is bounded in every sphere, is a topological quasi-rotation group. In particular, it is the topological rotation group whenever the period function is constant over the moving points.

1. To make clear the meaning of the theorem, let us denote by R a fixed euclidean three-space of reference, given in cylindrical coordinates: (r, θ, z). Let $F(r, z)$ be any positive continuous function defined for all z and all $r > 0$. For a fixed real number t, the correspondence

$$T_F(r, \theta, z; t): \qquad (r, \theta, z) \to (r, \theta + 2\pi[F(r, z)]t, z)$$

defines a homeomorphism of R into itself which we shall call a quasi-rotation. The totality of these mappings, for all real t, is a group and we call it a quasi-rotation group. In particular, when $F(r, z)$ is constant, it is the rotation group and is independent, topologically speaking, of the value of the constant. Now let H denote any homeomorphism of R into E. Corresponding to each t, for a fixed $F(r, z)$, H induces a homeomorphism of E into itself: their totality, for all t, will be called a *topological quasi-rotation group* of E. The proof of our theorem will consist in constructing a suitable function F and homeomorphism H.

2. In accordance with our premises, let $T(x; t)$ be a *one-parameter group* of homeomorphisms of E such that

* Presented to the Society, September 12, 1935; received by the editors June 12, 1935.

† National Research Fellow.

(i) $T(x; t)$ is continuous simultaneously in x and t;

(ii) to each solid sphere S in E there corresponds an integer N such that if $x \epsilon S$ there exists at least one $\bar{t} = \bar{t}(x)$ for which

$$T(x; \bar{t}) = x, \qquad 0 < \bar{t} < N.$$

For each x the set of points $T(x; t)$ is a continuous image of the set of real numbers. It will be called the *orbit* of x and designated by O_x. The set of real numbers t for which $T(x; t) = x$ is a closed subgroup (not the identity) of the real additive group, depending on x, and O_x is homeomorphic to the corresponding factor group. Therefore O_x is either a simple closed curve, the homeomorph of a circumference, or the single point x. When $O_x = x$, x will be called a *fixed point* and said to have the period zero. Otherwise x will be called a *moving point* and its period $p(x)$ will be the least positive number for which $T[x; p(x)] = x$. By condition (ii) the period function $p(x)$ is bounded in every sphere. By the group property it is constant over each orbit. The following lemma is quite easy.

LEMMA 1. *If the sequence of points x_n converges to a moving point x, and the sequence of numbers $p(x_n)$ to a number p, then p is a positive integral multiple of $p(x)$.*

As an immediate corollary $p(x)$ is lower semi-continuous.

3. **The orbit-space E^*.** If the point y is in O_x, there is a t, $0 \leqq t < p(x)$, such that $T(x; t) = y$. It is a trivial consequence of the group property that two orbits have no point in common unless they coincide. Further, if any sequence of points x_n converges to a point x, then

$$\overline{\sum O_{x_n}} \supset O_x$$

by condition (i), and

$$\overline{\sum O_{x_n}} - \sum O_{x_n} \subset O_x$$

in virtue of the boundedness of the period function. That is, $O_x = \lim O_{x_n}$. We may conclude that if the points x_n converge to the point at infinity, that is, properly diverge, then any sequence of points drawn from the corresponding orbits must properly diverge. It is convenient to adjoin to E the point at infinity, P_∞, thus obtaining a closed three-sphere $\bar{E}$. We shall extend $T(x; t)$ by the following definition: $T(P_\infty; t) = P_\infty$ for all t. In this way P_∞ becomes its own orbit and is a fixed point. The extended function remains continuous simultaneously and it is still true that if $x_n \to x$, then $O_{x_n} \to O_x$.

We can now construct the important auxiliary space E^* whose points are in 1–1 correspondence with the orbits of $\bar{E}$, and which we may suppose metrized.† The continuous single-valued correspondence of $\overline{E}$ into E^* we shall denote by L. If a is any point of E^*, the inverse $L^{-1}(a)$ is an orbit, and A is a closed set in E^* when $L^{-1}(A)$ is closed. Finally since E is compact, connected, and locally connected, E^* will have all of these properties. It will be proved later that E^* is *homeomorphic to a closed* 2-*cell*. We may now define for every a in E^* a function $p^*(a)$ which has as its value the constant value assumed by $p(x)$ on $L^{-1}(a)$. The function $p^*(a)$ is lower semi-continuous.

4. Let F denote the set of fixed points of $\overline{E}$, M the set of moving points, D the subset of M at which $p(x)$ is discontinuous, and C the subset of M where $p(x)$ is continuous. Then $\overline{E}=F+C+D$, these sets being mutually exclusive. It is clear that F is closed, and it is not vacuous since it certainly contains P_∞. We shall assume that F does not exhaust E, as otherwise the analysis of $T(x; t)$ is trivial. Then $M=C+D$ is open and not vacuous. It is clear that $E^*=L(F)+L(C)+L(D)$, the three sets being mutually exclusive.

Lemma 2. *The set C is open and not vacuous.*

This is an easy consequence of Lemma 1. There is no reason, a priori, why D should be vacuous, but we shall see later that it is.

The system of orbits contained in M is readily shown to be a regular family as defined by H. Whitney.‡ It is in fact a very special type of such families and we shall therefore be free to exploit the "cross sections" which he has introduced to obtain what shall here call a *true section*.

Definition. If A is a subset of E^*, any relatively closed subset K of $L^{-1}(A)$ which *cuts each orbit of $L^{-1}(A)$ exactly once* will be called a true section of $L^{-1}(A)$. It is clear that K will be homeomorphic to A.

By Lemma 2, $D+F$ is closed; it follows that $L(D+F)$ is closed and that its complement $L(C)$ is open in E^*.

Lemma 3. *If b is any point in $L(C)$ there is a neighborhood U of b such that $L^{-1}(U)$ contains a true section.*

Let U_1 be any neighborhood of b contained in $L(C)$. The orbits filling $L^{-1}(U_1)$ constitute a regular family, so that if q is any point of $L^{-1}(b)$ there exists a set S which is a *cross section through* q of the curves filling $L^{-1}(U_1)$ [in the sense of Whitney]. Let R_n be the closed sphere of center q and radius $1/n$. The set $R_n \cdot S$ is a cross section through q.§ It will now be shown that for some n each orbit contains at most one point of $R_n \cdot S$. Suppose, contrariwise, that in each set $R_n \cdot S$ there is a point q_n such that O_{q_n} cuts $R_n \cdot S$ in a second

† See Hausdorff, *Mengenlehre*, 1927, p. 145, for a definition of distance between closed sets.

‡ Annals of Mathematics, vol. 34 (1933), pp. 244–270.

§ We shall not repeat the definition and properties of the Whitney cross sections, except by implication.

point r_n. There is then for each n a number t_n such that $|t_n| \leqq \frac{1}{2} p(q_n)$ and such that $T(q_n; t_n) = r_n$. Since $p(x)$ is bounded in every bounded set, it may be assumed that $\{t_n\}$ converges to a unique limit t_0. The sequences $\{q_n\}$ and $\{r_n\}$ converge to q, so that $q = \lim r_n = \lim T(q_n; t_n) = T(q; t_0)$. On account of the continuity of $p(x)$ and the fact that $|t_n| \leqq \frac{1}{2} p(q_n)$ it follows that $t_0 = 0$. Consider now the arcs $c_n d_n$ made up of points $T(q_n; t) [-|t_n| < t < |t_n|]$. These arcs contain q_n and r_n. But if $q'q''$ is any arc of O_q containing q, and λ is any number, it is easy to see that for some n, there is an arc $c_n d_n$ contained in an arc of the λ-neighborhood (in Whitney's sense, loc. cit.) of $q'q''$. Since this is in contradiction with his cross section, we may conclude that there must exist an n_1 such that each orbit which cuts $R_{n_1} \cdot S$ cuts it exactly once.

The set $R_{n_1} \cdot S$ is mapped in a 1–1 manner on the set $L(R_{n_1} \cdot S)$. Because $R_{n_1} \cdot S$ is compact this mapping is a homeomorphism. From the properties of the Whitney section the set $L(R_{n_1} \cdot S)$ must contain all points in some neighborhood U of b. If K is the part of $R_{n_1} \cdot S$ which is mapped into U, K is a true section of $L^{-1}(U)$.

5. In this section we shall investigate the nature of the inverse images of certain simple sets in E^* preparatory to showing that E^* is a closed 2-cell.

LEMMA 4. *If $\alpha\beta$ is an arc in $L(C)$, then $L^{-1}(\alpha\beta)$ contains a true section and is homeomorphic to the product of a circle and a segment.*

About each point b of $\alpha\beta$ choose a neighborhood U such that $L^{-1}(U)$ contains a true section K. Any subarc $\alpha_1\beta_1$ of $\alpha\beta$ which lies in a single U is clearly such that $L^{-1}(\alpha_1\beta_1)$ contains a true section $\alpha_1'\beta_1'$. Since $\alpha\beta$ may be covered by a finite set of the U's mentioned above, there is on $\alpha\beta$ a set of non-overlapping arcs $(\alpha_0\alpha_1)$, $(\alpha_1\alpha_2)$, $\cdots$, $(\alpha_{n-1}\alpha_n)$ whose sum is $\alpha\beta$ and each of which is in a single U. Let $\alpha_{i-1}''\alpha_i'$ be a true section of $L^{-1}(\alpha_{i-1}\alpha_i)$. There is a real number t_1 such that $T(\alpha_1''; t_1) = \alpha_1'$. Consider the arc $\alpha_0''\alpha_1' + T(\alpha_1''\alpha_2'; t_1)$.† This arc is a true section of $L^{-1}(\alpha_0\alpha_1 + \alpha_1\alpha_2)$ obtained by turning the true section of $L^{-1}(\alpha_1\alpha_2)$ so that its first end point matches with the last end point of the true section of $L^{-1}(\alpha_0\alpha_1)$. By making successive turnings of this type, we obtain an arc $\alpha'\beta'$ which is a true section of $L^{-1}(\alpha\beta)$.

If I is the unit interval and C_1 is a circle of circumference 2π we can show that $L^{-1}(\alpha\beta)$ and the product-space $I\text{x}C_1$ are homeomorphic as follows. Let $h(x)$ be any homeomorphic mapping between $\alpha'\beta'$ and I. Suppose now that y is a point of $L^{-1}(\alpha\beta)$ which is on the orbit cutting $\alpha'\beta'$ in x and suppose that $T(x; t) = y$ where $0 \leqq t < p(x)$. Then because of the continuity of $p(x)$ in C, a homeomorphism between $L^{-1}(\alpha\beta)$ and $I\text{x}C_1$ is defined by the correspondence: $y \longleftrightarrow h(x)\text{x}2\pi t/p(x)$.

† This shall denote the set of all points x such that $x = T(x'; t_1)$ for some $x' \subset \alpha_1''\alpha_2'$.

LEMMA 5. *If the arc* $\alpha\beta$ *is in* $L(C)$ *except for the point* α *which is in* $L(F)$, *then* $L^{-1}(\alpha\beta)$ *has a true section and is a closed 2-cell.*

Express $\alpha\beta-\alpha$ as $\sum_{n=1}^{\infty}\sigma_n$ where the σ_n's are consecutive closed arcs, σ_n and σ_{n+1} abutting at a common end point β_n. Let S_n be a true section of $L^{-1}(\sigma_n)$. The end points b_1 and b_1' of S_1 and S_2 respectively, corresponding to β_1, lie on the same orbit. Choose a t^* so that $T(b_1'; t^*)=b_1$. The arc $S_2'=T(x; t^*)$, x ranging over S_2, is also a true section of $L^{-1}(\sigma_2)$, and S_1+S_2' is a true section of $L^{-1}(\sigma_1+\sigma_2)$. Continuing this indefinitely we obtain a true section K of $L^{-1}(\alpha\beta-\alpha)$. It is clear (compare §3) that if f denotes the fixed point $L^{-1}(\alpha)$, $K+f$ is closed and is the desired section. That $L^{-1}(\alpha\beta)$ is a 2-cell is now trivial.

LEMMA 6. *If* J *is a simple closed curve in* $L(C)$, *then* $L^{-1}(J)$ *contains a true section and is a torus.*

We express J as the sum of two arcs $\alpha\gamma\beta$ and $\alpha\gamma'\beta$. Let acb and $a'c'b'$ be true sections of $L^{-1}(\alpha\gamma\beta)$ and $L^{-1}(\alpha\gamma'\beta)$ respectively. Choose t and t' so that

$$T(a; t) = a', \qquad T(b; t') = b'.$$

Let h be a continuous function defined for all x of acb which has the value t at a and t' at b. The set of points

$$T[x; h(x)], \qquad x \text{ in } acb,$$

is a true section of $L^{-1}(\alpha\gamma\beta)$, and is an arc whose end points coincide with a' and b' respectively. Together with the arc $a'c'b'$ it gives the desired true section of $L^{-1}(J)$. That $L^{-1}(J)$ is a torus can now be seen by defining a homeomorphism similar to the one used in Lemma 4.

6. We shall devote this section to proving that D is vacuous. A certain type of set used below will first be described. Let B be the product of a circle and an interval, in other words a bounded cylinder. Let points of the base be specified by an angular coordinate θ $(0\leqq\theta\leqq 2\pi)$ and on one base of the cylinder identify sets of k points of the form $\theta+2n\pi/k$ $(n=0, 1, \cdots, k-1)$. Any homeomorph of this figure is called a twisted strip of index k.

LEMMA 7. *Let* $\alpha\beta$ *be an arc which, with the exception of* α, *lies in* $L(C)$, *and which is such that as* x *approaches* α *along* $\alpha\beta$, $p^*(x)$ *approaches* $kp^*(\alpha)$. *Then* $L^{-1}(\alpha\beta)$ *contains a true section and is a twisted strip of index* k.

We observe first that $L^{-1}(\alpha\beta)$ is locally connected. For let g be any point of this set, U_ϵ the open sphere about g of radius ϵ. There is an open arc of the orbit O_g which contains g and lies entirely in U_ϵ. Let $\delta<\epsilon$ be such that the sphere U_δ about g does not intersect O_g except in points of the open arc Let $\gamma=L(g)$. It is clear, by §3, that there is an arc σ, open in $\alpha\beta$ and containing γ, such that every orbit in $L^{-1}(\sigma)$ intersects U_δ. Let Q be the subset of $L^{-1}(\sigma)$ consisting of all points x such that $x=T(\bar{x};\bar{t})$, where $\bar{x}$ is some point of U_δ and $T(\bar{x};t)$ is in U_ϵ for every t, $0\leqq t\leqq \bar{t}$ if $t>0$, and for every t, $0\geqq t\geqq \bar{t}$ if $t<0$. It is now easy to see that Q is connected and open in $L^{-1}(\alpha\beta)$. Hence $L^{-1}(\alpha\beta)$ is locally connected.

Therefore if q is any point of $L^{-1}(\alpha)$ and S is a cross section through q of the regular family filling $L^{-1}(\alpha\beta)$, S is locally connected.† Then there must be an arc qr which is contained in S and which has q and no other point on $L^{-1}(\alpha)$.

It will now be shown that qr cuts no orbit more than once. For, if it did, it would contain a proper subarc $r'r$ with the same property. Now $r'r\subset L^{-1}(\alpha'\beta)$, where $\alpha'\beta$ is a proper subarc of $\alpha\beta$. In virtue of Lemma 4, a quite elementary argument will show that there exists a point v of $r'r$ and there exists an arbitrarily small arc σ on the orbit of v, containing v as inner point, such that on arbitrarily near orbits (possibly on O_v itself) there exist arcs arbitrarily near to σ (in the sense of Whitney) which meet $r'r$ (therefore S) in at least two distinct points. Hence the first condition that S be a cross section (Whitney, loc. cit.) fails at the point v. Therefore qr cuts all orbits at most once.

It follows that qr is mapped homeomorphically on a subset $\alpha\beta_1$ of $\alpha\beta$ and $\alpha\beta_1$ is an arc. If β_1 is different from β, there is an arc $\beta_1'\beta'$ which is a true section of $L^{-1}(\beta_1\beta)$. By turning $\beta_1'\beta'$ so that β_1' coincides with r we obtain an arc which we denote by qr' and which is a true section of $L^{-1}(\alpha\beta)$.

That the surface swept out by qr' under the action of the group is a twisted strip of index k may be seen as follows. Let Q be a cylinder which is the product of K and I, K being a circle of circumference 2π and I being a unit interval. Let y be any point of $L^{-1}(\alpha\beta)$ and denote by x' the point in which the orbit through y cuts qr'. Let t be such that $T(x';t)=y[0\leqq t<p(y)]$. Let $h(x')$ be a homeomorphism carrying qr' into I. Now consider the following correspondence. If y is in $L^{-1}(\alpha\beta-\alpha)$, y corresponds to $h(x')\mathrm{x}2\pi t/p(y)$. If y is in $L^{-1}(\alpha)$, y corresponds to $h(x')\mathrm{x}2\pi t/(kp(y))$. This determines a correspondence between $L^{-1}(\alpha\beta)$ and certain points of Q. However, when the proper identifications are made on Q to make it a twisted strip of index k, the above correspondence actually is a homeomorphism of $L^{-1}(\alpha\beta)$ and this twisted strip.

Principal Lemma 1. *The set D is vacuous; that is, $p(x)$ is continuous on the moving points.*

We shall show that $L(D)\subset E^*$ is vacuous. We know that $L(D)$ is closed

† Whitney, loc. cit.

in $L(M)$, hence $L(D)=P+G$ where P is perfect in $L(M)$ and G is countable. There are two cases:

(i) P is vacuous. Here $L(D)=G$ and contains an isolated point b. Since $L(F)$ is closed, there exists in E^* an arc $\beta' b$ which, except for b, belongs to $L(C)$ where $p^*(x)$ is continuous. Since D is, in this case, a one-dimensional F_σ,† it is well known that its complement in E is arcwise connected. Therefore if f^* denotes any point of $L(F)$, there is in E^* an arc $\beta' f^*$ which contains no point of $L(D)$. Let f be the first point of this arc which is in $L(F)$. In the sum of the arcs $b\beta'$ and $\beta' f$ there is an arc $b\beta f$ such that the open arc belongs to $L(C)$.

Now it is quite easy to see that, at an isolated point b in $L(D)$, $p^*(x)$ approaches a unique limit, say $kp^*(b)$, where by Lemma 1, k must be an integer. Then from previous lemmas, $L^{-1}(b\beta)$ is a twisted strip of index k, and $L^{-1}(\beta f)$ is a 2-cell. It is clear that $L^{-1}(b\beta f)$ is homeomorphic to a complex which contains no absolute cycle, but which is a cycle mod k. Such a configuration cannot exist in E, as can be seen from the Alexander Duality Theorem.‡ In this first case we have now arrived at a contradiction.

(ii) P is not vacuous. The function $p^*(a)$ considered relatively to P is lower semi-continuous and there must be a point b in P at which $p^*(a)$ is continuous§ considered relatively to P. Let O be an open set in $L(M)$ which contains b and which is such that if $a \epsilon O$, $p^*(a)<Np^*(b)$ where N is some fixed integer. Assume further that O is so chosen that if $a \epsilon O\cdot P$,

$$(1) \qquad | p^*(a) - p^*(b) | < \frac{p^*(b)}{6N}.$$

Let H_i^* be the set of all points a in $L(C)$ such that

$$(2) \qquad | ip^*(b) - p^*(a) | < \frac{p^*(b)}{6}.$$

Let $H_i=H_i^*\cdot O$. Assume now that O is chosen so small that it may be written as a sum of (mutually exclusive) sets as follows: $O=P\cdot O+G\cdot O+\sum_1^N H_i$. Because b is in $L(D)$ there must be an integer $k\geqq 2$ such that H_k is not vacuous. Let B denote all those points of the boundary of O_k which lie in O, where O_k is used to denote a component of H_k. Let x_0 be a point in $L^{-1}(O_k)$ and let x_1 and x_2 be two points on distinct orbits in $L^{-1}(O\cdot P)$. There are at least two such points because P is dense in itself. Join x_0 and x_1 by an arc x_0x_1 which is

† By the "Summensatz": Menger, *Dimensionstheorie*, p. 92.

‡ Alexander, these Transactions, vol. 23, p. 333; and Pontrjagin, Annals of Mathematics, vol. 35, p. 904.

§ Hausdorff, *Mengenlehre*, 1927, p. 255.

in $L^{-1}(O)$ and which intersects neither $L^{-1}(G)$ nor x_2 (the set to be avoided is a 1-dimensional F_σ). Join x_0 and x_2 by an arc x_0x_2 which is in $L^{-1}(O)$ and which intersects neither $L^{-1}(G)$ nor the arc previously constructed† (except of course at the point x_0). Let x_3 be the first point of $L^{-1}(P\cdot O)$ on x_0x_1 and let x_4 be the first point of $L^{-1}(P\cdot O)$ on x_0x_2. These points are in $L^{-1}(B)$. The set $L(x_0x_3)+L(x_0x_4)$ is connected and locally connected; let $L(x_3)=a_3$ and $L(x_4)=a_4$. There must be an arc a_3a_4 which is a subset of $L(x_0x_3)+L(x_0x_4)$. Except for a_3 and a_4 this arc lies entirely in O_k. The function $p^*(a)$ considered along the arc a_3a_4 must approach an integral multiple of $p^*(a_3)$, say $k_3p^*(a_3)$, as a approaches a_3 along the arc, and as a approaches a_4 along the arc, $p^*(a)$ must approach an integral multiple of $p^*(a_4)$, say $k_4p^*(a_4)$.

For a in the interior of the arc we must have, since $a\epsilon O_k\subset H_k\subset H_k^*$,

$$|\,kp^*(b) - p^*(a)\,| < \frac{p^*(b)}{6}. \tag{3}$$

Since a_i $(i=3, 4)$ is in $O\cdot P$ and since $k\leqq N$, (1) implies

$$|\,p^*(b) - p^*(a_i)\,| < \frac{p^*(b)}{6k}.$$

This inequality implies

$$|\,kp^*(b) - kp^*(a_i)\,| < \frac{p^*(b)}{6}. \tag{4}$$

From (3) and (4) we have the following:

$$|\,p^*(a) - kp^*(a_i)\,| < \frac{p^*(b)}{3}. \tag{5}$$

For a on the arc, $p^*(a)>p^*(b)$ and therefore (5) implies that k_3 and k_4 both equal k.

Let a_5 be some point on the interior of a_3a_4. The sets $L^{-1}(a_5a_3)$ and $L^{-1}(a_5a_4)$ are both twisted strips of index k. These strips combine to form a set which is a cycle modulo k but which contains no cycle in the absolute sense. Again by the Alexander Duality Theorem such a set cannot be imbedded in $\overline{E}$ and a contradiction has been reached. Therefore both P and G must be vacuous and the set $L(D)$ is vacuous.

7. It will assist us in analyzing F to construct a certain homeomorphism, defined as follows: For any point x, $H(x)$ is the point $T[x;\, p(x)/2]$. That $H(x)$ is a homeomorphism follows from the properties of $T(x;\, t)$ and the fact

† Again the set to be avoided is a 1-dimensional F_σ.

that $p(x)$ is continuous at moving points. If x_0 is a fixed point the continuity of $H(x)$ at x_0 follows even though $p(x)$ is not continuous at x_0 on account of the boundedness required of $p(x)$. The fixed points of $H(x)$ are identical with the fixed points of $T(x; t)$. The transformation $H(x)$ clearly has period 2.

LEMMA 8. *The set F is nowhere dense.*

The set F is closed and would contain inner points if the lemma were false. But if the fixed points of H contain inner points, H is the identity† and this case has been excluded from consideration.

LEMMA 9. *If O is an open connected set which is invariant under $T(x; t)$, then $O-O\cdot F$ is arcwise connected.*

Suppose that $O-O\cdot F=A+B$ where A and B are disjoined non-vacuous sets such that $A\overline{B}+\overline{A}B=0$. Since $O-O\cdot F$ is open, A and B must be open. The set O is transformed homeomorphically into itself by H, and furthermore $H(A)=A$, $H(B)=B$, and $H(O\cdot F)=O\cdot F$. A new transformation K of O into itself will now be defined. If $x\epsilon B+O\cdot F$, $K(x)=H(x)$; if $x\epsilon A$, $K(x)=x$. The transformation K is a homeomorphism of A into itself. But the set of fixed points of K contains inner points. Since K has period 2 and since O is locally euclidean, it follows from Newman's theorem that every point of O is fixed under K. This contradicts the definition of K and hence $O-O\cdot F$ must be connected. That it is arcwise connected is now immediate, since $O\cdot F$ is closed in O.

It follows at once from this lemma that M is connected.

LEMMA 10. *If q is any fixed point there is a connected open set O including q which has an arbitrarily small diameter and is invariant.*

Take a sphere containing q and let O be the set swept out by this sphere under the action of $T(x;t)$. The conditions on $p(x)$ imply that the diameter of O may be made small by properly choosing the sphere about q.

LEMMA 11. *Every fixed point is arcwise accessible from M.*

This is a consequence of the two preceding lemmas.

PRINCIPAL LEMMA 2. *The set $L(M)$ in E^* is homeomorphic to the euclidean plane.*

Let J denote any simple closed curve of $L(M)$. The set $L^{-1}(J)$ is a torus by Lemma 6, and $\overline{E}-L^{-1}(J)$ is the sum of two connected domains O_1 and O_2, say, such that $\overline{O}_i-O_i=L^{-1}(J)$, $i=1, 2$, by the Alexander Duality Theorem. Since $L^{-1}(J)$ is invariant, it is clear that O_1 and O_2 are invariant. By Lemma 9,

† See Newman, Quarterly Journal of Mathematics, vol. 2 (1931), p. 1.

$O_i - O_i \cdot F$ is connected. It is clear that $L(O_1 - O_1 \cdot F)$ has no point in common with $L(O_2 - O_2 \cdot F)$ and that $L(M) - J$ is equal to the sum of these two sets; that is, J separates $L(M)$. We have, in fact, shown the stronger result that J separates E^*. Now if G is any arc of J, $L^{-1}(G)$ is a closed proper subset of $L^{-1}(J)$. Therefore $\bar{E} - L^{-1}(G)$ is open and connected, and $\bar{E} - L^{-1}(G) - F$ is connected. But $L[\bar{E} - L^{-1}(G) - F] = L(M) - G$. Therefore G does not separate $L(M)$.

To complete the proof it is necessary to show† that one of the domains of $L(M) - J$ is compact in $L(M)$ and that the other is not. One of the domains, we may suppose it is O_2, contains P_∞. It is clear that $L(O_2 - O_2 \cdot F)$ is not compact in $L(M)$. Then we have left to show that O_1 does not contain points of F: in this event $L(O_1)$ *is* compact in $L(M)$. Since J is accessible‡ from each of its domains, if $L(O_1)$ contains a point of $L(F)$, E^* must contain an arc $a_1 a_0 a_2$ such that $a_1 + a_2$ is in $L(F)$, a_0 is in J, the open arc $a_1 a_0$ is in $L(O_1 - O_1 \cdot F)$, and the open arc $a_0 a_2$ is in $L(O_2 - O_2 \cdot F)$. The sets $L^{-1}(a_1 a_0)$ and $L^{-1}(a_2 a_0)$ are 2-cells by Lemma 5 and it is clear that their sum is a 2-sphere S_2 which has in common with the torus $L^{-1}(J)$ the simple closed curve $L^{-1}(a_0)$. By the Duality Theorem, S_2 separates E into exactly two open connected domains G_1 and G_2. Now $L^{-1}(J - a_0)$ is connected and must lie entirely in one or the other of these sets, say in G_1. This implies that there cannot be points of G_2 in both O_1 and O_2. Since the points of $L^{-1}(a_1)$ and of $L^{-1}(a_2)$ (points of S_2) must be accessible from G_2, this is a contradiction.

8. We shall now prove the following lemma:

LEMMA 12. *If $\alpha\beta$ is an arc in $L(M)$ and if qr and $q'r'$ are two true sections of $L^{-1}(\alpha\beta)$, there exists a real continuous function $h(x)$ defined everywhere on qr such that $T[x; h(x)]$ is a homeomorphic mapping of qr on $q'r'$.*

Let a denote any point of $\alpha\beta$, x and x' the corresponding points of qr and $q'r'$. It is clear that the correspondence $x' \longleftrightarrow x$ engenders a homeomorphism.

For a definite point x let t be a real number such that $T(x; t) = x'$. Let x_n denote any sequence of points of qr converging to x. Assign to each x_n a number t_n, $|t_n| \leqq \frac{1}{2}p(x_n)$, such that $T(x_n; t + t_n) = x_n'$. At least one such number exists and at most two. We may suppose our sequence of points such that t_n converges to a limit t'. Then $T(x_n; t + t_n) \to T(x; t + t')$ which is equal to $T(x'; t')$. On the other hand $x_n' \to x'$ so that $T(x; t') = x'$, from which we may conclude that $t' = mp(x')$ where m is some integer or zero. Since $p(x)$ is continuous at x', t' cannot exceed $\frac{1}{2}p(x')$ in absolute value, and m must be zero. It follows that for almost all x_n only one choice of t_n is possible and we have

† L. Zippin, *On continuous curves*, American Journal of Mathematics, vol. 52 (1930), pp. 331–350.
‡ L. Zippin, loc. cit.

seen that this choice must be continuous at x. It can now be seen that there must exist a sufficiently small subarc $q''r''$ of qr containing x as an inner point such that a real continuous function $h''(x)$ may be defined on it for which $T[x; h''(x)]$ maps $q''r''$ homeomorphically on the corresponding subarc of $q'r'$. We need only choose an arc so small that the numbers t_n assigned above are actually *less* in absolute value than one-half the corresponding period.

The arc qr may be expressed as a sum of non-overlapping subarcs $q_0q_1, q_1q_2, \cdots, q_{k-1}q_k$, such that there is defined on $q_{i-1}q_i$ a function $h_i(x)$ of the required sort. Now the values of h_1 and h_2 at q_1 can differ only by an integral multiple m of the period at that point. Then $h_2' = h_2 + mp(x)$ is a new function defined on q_1q_2 which is of the required type, and this new function agrees with h_1 at q_1. The function which is equal to h_1 on q_0q_1 and equal to h_2' on q_1q_2 is of the required type over $q_0q_1+q_1q_2$. We can continue in this manner, and after k steps we arrive at a function defined over qr which satisfies the lemma.

Lemma 13. *If S is a closed 2-cell in $L(M)$ there exists a true section of $L^{-1}(S)$.*

We can express S as the sum of a finite number of closed 2-cells $A_1, \cdots, A_n$, of small diameter such that $A^i = \sum_1^i A_j$ is itself a closed 2-cell and the intersection of A^k and A_{k+1} is an arc J_{k+1}. Since $L^{-1}(S)$ admits true sections locally by Lemma 3, it may be assumed that to each A_i there corresponds a true section B_i of $L^{-1}(A_i)$.

Let $B^1 = B_1$, and suppose we have constructed a true section B^k of $L^{-1}(A^k)$. Let J be the image of J_{k+1} in B^k and J' its image in B_{k+1}. Let $h(x)$ be the function defined on J' in the preceding lemma, and let $H(x)$ be any continuous extension of this function over all of A_k. Then $B^{k+1} = B^k + T[x; H(x)]$ where x is allowed to vary over B_{k+1}. This set is seen to be a true section of $L^{-1}(A^{k+1})$. The induction is complete and Lemma 13 is proved.

It follows from this lemma that $L^{-1}(S)$ can be mapped homeomorphically into the product of a circular disk D and a circle C in such a way that the path curves of $L^{-1}(S)$ are carried into the sets $d \times C$ where $d \in D$. Hence if S is a closed 2-cell in M and a is a point of S, $L^{-1}(a)$ cannot bound in $L^{-1}(S)$.

9. The fact just proved will be useful in the following lemma:

Lemma 14. *If a is a point of $L(M)$, $L^{-1}(a)$ cannot bound in M.*†

If $L^{-1}(a)$ bounds in M, it must bound a singular 2-chain K which is contained in M. The transformation L carries K into a closed compact subset of $L(M)$ which we may suppose to be contained in a closed 2-cell S where

† This is very similar to a theorem due to Seifert, Acta Mathematica, vol. 60, p. 224.

$S \subset L(M)$. But now $L^{-1}(a)$ bounds in $L^{-1}(S)$, which from the remark above cannot happen.

Now since any simple closed curve bounds in $\bar{E}$ even after the removal of a finite number of points, it is a corollary to the lemma that F is an *infinite* set.

LEMMA 15. *For every point z in F, $F-z$ is connected.*

If z is a point for which the lemma is false, F may be expressed as the sum of two closed (in F) sets X and Y whose intersection is a subset of z (possibly vacuous). If x and y denote points of X and Y respectively and if $L(x)=a$ and $L(y)=b$, there is an arc ab in E^* which contains no points of $L(F)$ other than a and b. This follows from the accessibility of points of F. Now $L^{-1}(ab)$ is a 2-sphere and bounds in $\bar{E}-z$ by the Alexander Duality Theorem. On the other hand if a_1 is an inner point of ab, $L^{-1}(a_1)$ bounds in the complement of X and in the complement of Y. This situation, however, contradicts corollary W^i of Alexander (loc. cit.), and this contradiction establishes the lemma.

Since F contains infinitely many points and is connected on the removal of any one of them, it must be connected.

PRINCIPAL LEMMA 3. *$L(\bar{E})$ is homeomorphic to a closed half-plane.*

If we let x and y denote any two distinct points of F, and if a, b, and ab are used as in the preceding lemma, the 2-sphere $L^{-1}(ab)$ separates E. Since it is an invariant surface we can conclude that the arc ab separates E^*. A proper subset of this arc gives rise to a proper subset of $L^{-1}(ab)$ and this does not separate E. We conclude that the arc ab separates E^* *irreducibly*.† But it is further clear that each domain of $E-L^{-1}(ab)$ contains at least one orbit, and since this bounds in its domain, we see that each domain must contain fixed points. Therefore the points x and y must separate the set F, and the points a and b must separate $L(F)$. Therefore $L(F)$ is a continuum without cut points which is separated by every pair of its points. It follows that $L(F)$ must be a simple closed curve.‡ Finally we have the fact that E^* is a closed 2-cell with $L(F)$ as its boundary.

It is clear that $E^*-L(P_\infty)$ which is identically $L(E)$ is a closed half-plane, the points $L(F)$ corresponding to the boundary line.

10. We are in a position to bring our entire argument to a close. We shall choose a Cartesian coordinate system (z, r) in $L(E)$, $-\infty<z<\infty$, $0\leqq r<\infty$, so that $L(F)$ is the z-axis, $r=0$. This is possible by the preceding

---

† L. Zippin, *Characterization of the closed 2-cell*, American Journal of Mathematics, vol. 55 (1933), pp. 207–217.

‡ R. L. Wilder, *Concerning simple continuous curves*, American Journal of Mathematics, vol. 53 (1931), pp. 39–55; Corollary, p. 48.

lemma. Now $L(M)$ is the point set $r>0$. We can obviously express this set as the sum of a countable number of "rectangles" such that the sum of any finite number of consecutive rectangles is a closed 2-cell. Corresponding to each of these there is a true section in M, by Lemma 13, and these can be "fitted" to each other by the construction in that lemma, where the induction must be carried indefinitely. Therefore M possesses a true section M', and M' is homeomorphic to $L(M)$. The homeomorphism is given by the function L. The function L is certainly a homeomorphism between F and $L(F)$, since each point of F is its own orbit. Then it is clear that L maps $M'+F$ homeomorphically into $L(E)$. For if a sequence of points of $L(M)$ converges to a point of $L(F)$, the corresponding orbits must converge to the fixed point in E, and therefore the corresponding points of M', which lie on these orbits, must converge to it.

We may now transfer to $E''=M'+F$ the coordinate system (z, r). We shall extend this to a cylindrical coordinate system in all of E. Consider now an arbitrary point x of E, our original three-space. It belongs to an orbit O_x which intersects E'' in a single point, say x'', with coordinates (z'', r''), say. If $r''=0$, x'' is a fixed point and $x=x''$. Then the "θ-coordinate" of x shall be indeterminate. If $r''>0$, there is a unique t'', $0\leqq t''<p(x'')=p(x)$, such that $T(x'', t'')=x$. The θ coordinate of x shall be $2\pi t''/p(x'')$. Conversely, it is clear that any three numbers z, θ, r,

$$-\infty<z<\infty, \qquad 0\leqq\theta\leqq 2\pi,$$

determine a unique point of E. With this coordinate system established in E we may take E to be the "reference-space" R of §1, and the homeomorphism of that section to be identity. The function $F(z, r)$ is now $1/p(z, r)$, $p(z, r)$ being the value of the period function at the point (z, r) of E''. We have seen that this function is independent of θ and is continuous at the moving points, i.e., points for which $r>0$.

Princeton University, and
Institute for Advanced Study,
Princeton, N. J.